

RNGSSELIB: Program library for random number generation, SSE2 realization

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Abstract

The library RNGSSELIB for random number generators (RNGs) based upon the SSE2 command set is presented. The library contains realization of a number of modern and most reliable generators. Usage of SSE2 command set allows to substantially improve performance of the generators. Three new RNG realizations are also constructed. We present detailed analysis of the speed depending on compiler usage and associated optimization level, as well as results of extensive statistical testing for all generators using available test packages. Fast SSE implementations produce exactly the same output sequence as the original algorithms.

PROGRAM SUMMARY

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Programming language: C

Computer: PC

Operating system: UNIX, Windows

RAM: 1 Mbytes

Number of processors used: 1

Supplementary material:

Keywords: Statistical methods, Monte Carlo, Random numbers, Pseudorandom numbers, Random number generation, Streaming SIMD Extensions

Classification: 4.13 Statistical Methods

External routines/libraries:

Subprograms used:

Nature of problem: Any calculation requiring uniform pseudorandom number generator, in particular, Monte Carlo calculations.

Solution method: The library contains realization of a number of modern and reliable generators: **mt19937**, **mr32k3a** and **lfsr113**. Also new realizations for the method based on parallel evolution of an ensemble of dynamical systems are constructed: **GM19**, **GM31** and **GM61**. The library contains both usual

realizations and realizations based on SSE command set. Usage of SSE commands allows to substantially improve performance of all generators.

Restrictions: For SSE realizations of the generators, Intel or AMD CPU supporting SSE2 command set is required. In order to use the realization **lfsr113sse**, CPU must support SSE4 command set.

Unusual features:

Additional comments:

Running time: Running time is of the order of 20 sec for generating 10^9 pseudorandom numbers with a PC based on Intel Core i7-940 CPU. Running time is analysed in detail in Sec. 5 of the paper.

1. Introduction

Pseudo random numbers, generated recursively by deterministic rules, represent one of important ingredients of numerical simulations widely used in physics and material science [1]. Design of the random number generators (RNG) producing pseudo random numbers which approximate “true randomness” [2] is a great challenge for the computational physics and computer science in general.

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High speed and statistical robustness are the most important characteristics needed for good RNG. Also, long period length of the generated sequences is among the most desired features.

In the paper we present a library of modern and most reliable RNGs known today. Namely, the realizations of the modern generators MT19937, MRG32k3a, LFSR113, GM19SSE, GM31SSE and GM61SSE are presented. MT19937 is the 2002 version of the Mersenne Twister generator of Matsumoto and Nishimura [3], which is based on the recent generalizations to the GFSR method. MRG32k3a is the combined multiple recursive generator proposed in [4], and LFSR113 is the combined Tausworthe generator of L'Ecuyer [5]. Each of the generators GM19SSE, GM31SSE and GM61SSE is based on an ensemble of sequences generated by multiple recursive method. The method is closely related to the method of random number generation based on evolution of the ensemble of the chaotic dynamical systems [6]. We introduce three RNG realizations based on the method: GM19SSE, GM31SSE and GM61SSE.

An important feature of our library is the ability to speed up the RNG generation using Streaming SIMD Extensions 2 (SSE2) technology, introduced in Intel Pentium 4 processors in 2001 [7]. AMD added SSE2 support to their processors with the introduction of their Opteron and Athlon64 ranges of 64-bit CPUs in 2003 [8]. All of Intel and AMD processors, fabricated later than 2003, support SSE2 instruction set. SSE2 technology allows to use 128-bit XMM-registers effectively and to accelerate computations. A similar technique was previously used for implementing some RNGs [9, 6]. In the present work we demonstrate that this approach can essentially speed up most RNGs.

Program codes for the generators and proper initializations can be found in [10]. We would like to stress that our SSE implementations of algorithms produce exactly the same output sequence as the original algorithms. We would appreciate any comments on user experiences.

2. Classical and modern random number generators

The most widely used RNGs can be divided into two classes. The first class is represented by linear congruential generator (LCG), and the second by shift register (SR), or Tausworthe generator.

Linear Congruential Generators (LCGs) are the best-known and (still) most widely available RNGs in use today. An example of the realization of an LCG generator is the UNIX `rand` generator $y_n = (1103515245 y_{n-1} + 12345) \pmod{2^{31}}$. The practical recommendation is that LCGs should be avoided for applications dealing with the geometric behavior of random vectors in high dimensions because of the bad geometric structure of the vectors that they produce [2, 11]. Another problem is that classical LCGs have relatively small period lengths. Also, the lower-order bits of the generated sequence have a far shorter period than the sequence as a whole if the modulo is set to a power of 2. An immediate generalization of LCG is Multiple Recursive Generator (MRG).

Generalized Feedback Shift Register (GFSR) sequences are widely used in many areas of computational and simulational physics [12, 13]. These RNGs are quite fast and possess huge periods given a proper choice of the underlying primitive trinomial. This makes them particularly well suited for applications that require many pseudorandom numbers. But several flaws have been observed in the statistical properties of these generators, which can result in systematic errors in Monte Carlo simulations. Typical examples include the Wolff single cluster algorithm simulations for the 2D Ising model [14] and 3D Ising model [15], random and self-avoiding walks [16, 17], and the 3D Blume-Capel model using local Metropolis updating [18].

Modern RNGs are modifications and generalizations to the LCG and GFSR methods, they have much better periodic and statistical properties [19]. In this work we discuss properties, efficient realizations and statistical tests for some of them.

2.1. Mersenne Twister random number generator MT19937

Algorithm MT19937 is a Mersenne Twister (MT) generator by Matsumoto and Nishimura [3]. In a sense, MT algorithm represents a modified and twisted GFSR generator. MT generates the vectors of word size by the recurrence

$$\mathbf{x}_{k+n} := \mathbf{x}_{k+m} \oplus (\mathbf{x}_k^u | \mathbf{x}_{k+1}^l)A. \quad (1)$$

For MT19937 values of parameters are chosen as follows: $n = 624$, $m = 397$, $(\mathbf{x}_k^u | \mathbf{x}_{k+1}^l)$ is the 32-bit integer obtained by concatenating one upper bit of \mathbf{x}_k and 31 lower bits of (\mathbf{x}_{k+1}^l) , and \oplus is exclusive or operation. A is a companion matrix:

$\mathbf{x}A = \text{shiftright}(\mathbf{x})$ if the least significant bit of \mathbf{x} is 0, otherwise $\mathbf{x}A = \text{shiftright}(\mathbf{x}) \oplus \mathbf{a}$, where vector \mathbf{a} consists of 32 bits and represents 32-bit integer constant 2567483615. MT19937 is a generator with a very long period $2^{19937} - 1$, and it provides 623-dimensional equidistribution of pseudorandom numbers up to 32-bit accuracy. As for most generators, Mersenne Twister is sensitive to poor initialization, so proper initialization is very important.

2.2. Combined Multiple Recursive Generator MRG32k3a

Algorithm MRG32k3a represents a Combined Multiple Recursive Generator (CMRG) found in [4] by means of extensive search of parameters for such generator. MRG32k3a algorithm combines two MRGs (see right column in Table 3). The state of the first MRG is described by variables $\mathbf{x}0, \mathbf{x}1, \mathbf{x}2$, whereas the variables $\mathbf{y}0, \mathbf{y}1, \mathbf{y}2$ represent the state of the second MRG. At each step, the MRG states are shifted as shown in the right column of Table 3, where the calculated values of $\mathbf{p}1$ and $\mathbf{p}2$ are the new outputs of the MRGs.

2.3. Combined Tausworthe generator LFSR113

Another modern generator is a combined Tausworthe generator, i.e., a combined GFSR sequence. In [5] an extensive search for good parameters for such generator is carried out, and, as a result, an algorithm LFSR113 for such generator is presented. Right column in Table 4 describes the algorithm. The state of the generator is described by variables $\mathbf{z}1, \mathbf{z}2, \mathbf{z}3, \mathbf{z}4$.

2.4. Generators GM19, GM31, and GM61 based on the ensemble of MRGs

Finally, we include in the library improved versions of the generators GM19 and GM31 worked out in [6], and also the new realization GM61. It is suggested in [6] to construct RNGs based on an ensemble of hyperbolic automorphisms of the unit two-dimensional torus,

$$\begin{pmatrix} x_i^{(n)} \\ y_i^{(n)} \end{pmatrix} = M \begin{pmatrix} x_i^{(n-1)} \\ y_i^{(n-1)} \end{pmatrix} \pmod{g}, \quad (2)$$

where $i = 0, 1, \dots, (s-1)$, the elements of two-dimensional matrix M are integers, $\det M = 1$, and $|\text{Tr}M| > 2$. The output of the generator is defined as

$$a^{(n)} = \sum_{i=0}^{s-1} \lfloor 2x_i^{(n)} / g \rfloor \cdot 2^i. \quad (3)$$

Dynamical system (2) is widely known under name cat map (there are two reasons for this terminology: first, CAT is an acronym for Continuous Automorphism of the Torus; second, the chaotic behavior of these maps is traditionally described by showing the result of their action on the face of the cat [20]). The main properties of the generator based on cat maps can be predicted theoretically because there is close relation between the periodic properties of cat maps and arithmetical properties of algebraic numbers [6]. The basic recurrence (2), representing an evolution of a simple nonlinear dynamical system on a discrete $g \times g$ lattice, is equivalent to linear recurrence modulo g with the same characteristic polynomial $f(x) = x^2 - kx + q$ as that of the matrix M :

$$x^{(n)} = kx^{(n-1)} - qx^{(n-2)} \pmod{g}, \quad (4)$$

$$y^{(n)} = ky^{(n-1)} - qy^{(n-2)} \pmod{g}, \quad (5)$$

where $k = \text{Tr}M$, $q = \det M = 1$ [21, 22, 6]. Therefore, in a sense, the method represents a modified and generalized MRG, where only part of the information of a generator state influences the generator output.

Table 1: Parameters of the generators GM19, GM31 and GM61.

Generator	k	q	g	Period
GM19	15	28	$2^{19} - 1$	$2.7 \cdot 10^{11}$
GM31	11	14	$2^{31} - 1$	$4.6 \cdot 10^{18}$
GM61	24	74	$2^{61} - 1$	$5.3 \cdot 10^{36}$

The modified RNG based on the same idea, where lattice parameter g is chosen to be Mersenne prime, and the characteristic polynomial $f(x) = x^2 - kx + q$ is chosen to be primitive over \mathbb{Z}_g , possess very good statistical and periodic properties. In this case the basic recurrence is closely related to so-called matrix generator of pseudorandom numbers studied in [2, 21, 23]. Primitivity of the characteristic polynomial guarantees maximal possible period $g^2 - 1$ of the output sequence, also it is required for the primitivity of $f(x)$ that $q \neq 1$. Table 1 shows parameter values chosen for generators GM19, GM31 and GM61. For generating random numbers it is convenient to employ the linear recurrence (4). As above, the output function is defined with (3), i.e. each bit of the output corresponds to its own recurrence, and $s = 32$ recurrences are calculated in parallel.

There is an easy algorithm to calculate $x^{(n)}$ in (4) very quickly from $x^{(0)}$ and $x^{(1)}$ for any large

n . Indeed, if $x^{(2n)} = k_n x^{(n)} - q_n x^{(0)} \pmod{g}$, then $x^{(4n)} = (k_n^2 - 2q_n)x^{(2n)} - q_n^2 x^{(0)} \pmod{g}$. As was mentioned already in [6], this helps to initialize the generator. To initialize all 32 recurrences, the following initial conditions are used: $x_i^{(0)} = x^{(iA)}$, $x_i^{(1)} = x^{(iA+1)}$, $i = 0, 1, \dots, 31$. Here A is a value of the order of $(p^2-1)/32$, which is not very close to a divisor of $p^2 - 1$ or to a large power of 2.

3. SSE-realizations of RNGs

Modern Intel and AMD processors support SSE2 instruction set. SSE2 is acronym for Streaming SIMD (Single Instruction Multiple Data) Extensions 2. SSE2 realization of instruction set allows user to perform the same operation in parallel on four 32-bit numbers placed in 128-bit XMM registers, or on two 64-bit numbers [7]. It is known that proper programming of 32-bit RNG using SSE command set may essentially decrease computation time needed for generating random number. For example, in paper [9] SSE realization of the combined RNG based on two Shift Register generators with shift register lengths 9689 and 4423 accelerated random number generation by factor 3.5.

In this section we describe realization of random number generators discussed in the previous section using SSE2 command set. These effective realizations are equivalent to classical algorithms discussed in previous section, i.e. all output values are the same. For example, MT19937SSE algorithm produces the same output values as well-known MT19937 introduced in [3], in contrast to other Mersenne Twister SSE algorithms introduced in [24].

Program codes for all realizations and examples of practical implementations, including proper initialization, can be found in [10].

3.1. SSE2 based MT19937

In Table 2 we present main ideas of our version for MT19937 algorithm (see Sec.2.1), based on SSE2 command set. The whole array of 624 doubleword integers is divided into fours of integers, and four recurrent relations (1) are calculated in parallel.

3.2. SSE2 based MRG32k3a

Right column in Table 3 shows usual MRG32k3a algorithm (see Sec. 2.2). The recurrent relations are: $x_{i+3} = ax_i + bx_{i+1} \pmod{m_1}$ and $y_{i+3} = cy_i + dy_{i+2} \pmod{m_2}$, where $a = -810728$, $b =$

Table 2: SSE2 realization for the main part of algorithm MT19937SSE (tempering is omitted in the table).

asm("movaps	2500(%0),%%xmm0\n" \
"movaps	4(%0),%%xmm1\n" \
"pand	upper_mask,%%xmm0\n" \
"pand	lower_mask,%%xmm1\n" \
"por	%%xmm0,%%xmm1\n" \
"movaps	%%xmm1,%%xmm0\n" \
"psrld	\$1,%%xmm1\n" \
"pand	and1_mask,%%xmm0\n" \
"pcmpgtd	zero_mask,%%xmm0\n" \
"pand	andA_mask,%%xmm0\n" \
"pxor	%%xmm1,%%xmm0\n" \
"movaps	1588(%0),%%xmm1\n" \
"cpl	\$227,%2\n" \
"j1	MyL1%=\n" \
"movaps	-908(%0),%%xmm1\n" \
"MyL1%=: pxor	%%xmm1,%%xmm0\n" \
"movups	%%xmm0,(%0)\n" \
"movaps	%%xmm0,2500(%0)\n" \
": : "r"	(mt+i), "r"(consts), "r"(i));

1403580, $c = -1370589$, $d = 527612$. Therefore, $x_{i+4} = ax_1 + bx_2 \pmod{m_1}$ and $x_{i+6} = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 \pmod{m_1}$, $y_{i+4} = \beta_0 y_0 + \beta_1 y_1 + \beta_2 y_2 \pmod{m_2}$, $y_{i+6} = \gamma_0 y_0 + \gamma_1 y_1 + \gamma_2 y_2 \pmod{m_2}$, where constants $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1$ and γ_2 can be easily calculated. This allows to calculate x_4, x_5, x_6 and x_7 from x_0, x_1, x_2 , and also y_4, y_5, y_6 and y_7 from y_0, y_1, y_2 . Calculation of fours of numbers was found to be very effective. Main parts of the SSE-realization of the MRG32k3a algorithm are shown in the left column in Table 3.

Since there are no SSE commands for taking modulo in XMM registers, we use the relation $z \equiv (f \cdot N + g) \pmod{m}$ for $z = f \cdot 2^{32} + g$ and $N = 2^{32} - m$. We calculate z_1, z_2, z_3 using formulas $z_0 = f_0 \cdot 2^{32} + g_0$, $z_1 = f_0 \cdot N + g_0 = f_1 \cdot 2^{32} + g_1$, $z_2 = f_1 \cdot N + g_1 = f_2 \cdot 2^{32} + g_2$, if $z_2 < m$ then $z_3 = z_2$ else $z_3 = z_2 - m$. This results in our case in $z_3 < m$. SSE commands admit simultaneous calculation of $x_4 \pmod{m_1}$ and $x_5 \pmod{m_2}$. Finally the RNG output is calculated as $p_1 - p_2 + m_1$ for $p_1 \leq p_2$ and $p_1 - p_2$ for $p_1 > p_2$.

3.3. SSE4 based LFSR113

Our realizations of LFSR113 and LFSR113SSE (see Sec.2.3) are presented in Table 4. The SSE

Table 6: Example of using the realization MRG32K3A-SSE.

```
#include<stdio.h>
#include"mrg32k3asse.h"

int main(){
    unsigned i; mrg32k3asse_state state;
    mrg32k3asse_SetState(&state);
    mrg32k3asse_genInit(123,123,123,123,123,123);
    mrg32k3asse_genPrintState();
    for(i=0;i<999999999;i++) mrg32k3asse_genRand();
    printf("10^8 Numbers Generated\n");
    printf("Next Output Value: %u\n",
        mrg32k3asse_genRand());
    mrg32k3asse_genPrintState();
    return 0;
}
```

realization utilizes `pmulld` and `pblendw` processor instructions of SSE4 command set. Therefore, the realization LFSR113SSE requires CPU supporting SSE4, for example, a processor with Intel Core microarchitecture. The realization works *only* for Intel processors and is not valid for AMD processors. For shifting four integers of an XMM register to the left by different numbers of bits, we use the SSE4 instruction `pmulld`. For shifting to the right by different numbers of bits, we use SSE4 command `pblendw`. The details of the algorithms are shown in Table 4.

3.4. SSE2 based GM19, GM31 and GM61

Table 5 illustrates the key ideas for speeding up algorithms GM19, GM31 and GM61 (see Sec. 2.4) using SSE2 command set. Actions of the fast SSE2 algorithms shown in the left column are equivalent to actions of the slow algorithms shown in the right column.

4. How to use the library. Function call interface.

4.1. Realization MRG32K3A-SSE

Table 6 shows the example of using the MRG32K3A-SSE realization in ANSI C language. Table 7 shows header file names for the generators. Using the realization requires including the header file `mrg32k3asse.h` in the code.

The function `mrg32k3asse_SetState(...)` in Table 6 saves the pointer to the structure `state` in order that the other functions use it as a pointer to

the structure that keeps the generator state. Such function should be executed prior to using any other functions that manipulate with a generator state. It can also be used in order to switch between using several RNGs of the same type. Table 8 describes the function call interface.

The function `mrg32k3asse_genInit(...)` in Table 6 initializes the generator state. Table 9 shows details for the initialization function. The values of `x0`, `x1`, `x2`, `y0`, `y1`, `y2` described in Sec. 2.2 and Table 3 are required to initialize any version of the generator MRG32K3A.

The function `mrg32k3asse_genRand()` in Table 6 generates a pseudorandom 32-bit integer and makes respective transformation to the generator state. Detailed interface for this function is presented in Table 7.

The function `mrg32k3asse_genRand4(unsigned* arr)` generates block of four pseudorandom 32-bit integers and puts them in `arr[0]`, `arr[1]`, `arr[2]` and `arr[3]`. The function `mrg32k3asse_genRand32(unsigned* arr)` generates block of 32 pseudorandom 32-bit integers and puts them in `arr[0]`, `arr[1]`, ..., `arr[31]`. Detailed interface for the functions is presented in Table 7. Generation of blocks of numbers is further discussed in Sec. 5 and Tables 13 and 14.

The function `mrg32k3asse_genPrintState()` prints out the generator state. The function call interface is shown in Table 8.

4.2. Realization MRG32K3A

The similar example as in Table 6 for the realization MRG32K3A, which is exact reproduction of the algorithm published in [4] and is based on usual command set, can be found in the file `mrg32k3a.c` in [10].

The header file for the realization is `mrg32k3a.h`, as it is shown in Table 7.

Table 8 describes the function call interface for the function `mrg32k3a_SetState(...)` that saves a pointer to the generator state.

Table 9 shows details for the initialization functions `mrg32k3a_genInit(...)`. The values of `x0`, `x1`, `x2`, `y0`, `y1`, `y2` described in Sec. 2.2 and Table 3 are required to initialize the generator MRG32K3A.

The function `mrg32k3a_genRand()` generates a pseudorandom floating point number between 0 and 1 and makes respective transformation to the generator state. The function call interface is presented in Table 7.

Table 3: Realizations MRG32K3aSSE (left column) and MRG32k3a (right column) producing equivalent outputs.

main parts of SSE2-based MRG32k3aSSE	MRG32k3a-L with x86-64 command set
<pre> unsigned s[16] __attribute__((aligned(16))); // { x0,0,x1,0, x2,0,x3,0, y0,0,y1,0, y2,0,y3,0 } unsigned consts[72] __attribute__((aligned(16))) = { x4ap,0,x4ap,0, x4bm,0,x4bm,0, x4cp,0,x4cp,0, x4addl,x4addh,x4addl,x4addh, x6ap,0,x6ap,0, x6bp,0,x6bp,0, x6cm,0,x6cm,0, x6addl,x6addh,x6addl,x6addh, m1,0,m1,0, [...] }; // Calculating x4 and x5 from x1 and x2: asm("movups 8(%0),%%xmm1\n" "movaps 16(%0),%%xmm2\n" "pmuludq 16(%1),%%xmm1\n" "pmuludq 32(%1),%%xmm2\n" "paddq 48(%1),%%xmm2\n" "psubq %%xmm1,%%xmm2\n" "movaps %%xmm2,%%xmm3\n" "psrlq \$32,%%xmm3\n" "pmuludq 128(%1),%%xmm3\n" "psubq %%xmm3,%%xmm2\n" "movaps %%xmm2,%%xmm3\n" "psrlq \$32,%%xmm3\n" "pmuludq 128(%1),%%xmm3\n" "psubq %%xmm3,%%xmm2\n" "psubq 128(%1),%%xmm2\n" "movaps %%xmm2,%%xmm3\n" "psrad \$31,%%xmm3\n" "psrlq \$32,%%xmm3\n" "pand 128(%1),%%xmm3\n" "paddq %%xmm3,%%xmm2\n" "movaps %%xmm2,(%0)\n" "":"r"(s),"r"(consts)); // Calculating (x0-y0) (mod m1): asm("movaps (%0),%%xmm1\n" "psubq 32(%0),%%xmm1\n" "movaps %%xmm1,%%xmm4\n" "psrad \$31,%%xmm4\n" "psrlq \$32,%%xmm4\n" "pand 128(%1),%%xmm4\n" "paddq %%xmm4,%%xmm1\n" "":"r"(s),"r"(consts); </pre>	<pre> const long long add1=9007203111403311U; const long long add2=9007202867859652U; // here m1 divides add1, m2 divides add2 const long long m1= 4294967087U; const long long m2= 4294944443U; const long long a = -810728; const long long b = 1403580; const long long c = -1370589; const long long d = 527612; long long x0, x1, x2, y0, y1, y2; unsigned mrg32k3a (){ long k; long long p1, p2; p1 = (add1 + b*x1 + a*x0) % m1; x0 = x1; x1 = x2; x2 = p1; p2 = (add2 + d*y2 + c*y0) % m2; y0 = y1; y1 = y2; y2 = p2; if (p1 <= p2) return (p1 - p2 + m1); else return (p1 - p2); } </pre>

Table 4: Algorithms LFSR113 and LFSR113SSE

SSE4 based LFSR113SSE	LFSR113 based on usual instruction set
<pre> unsigned z[4] __attribute__((aligned(16))) = {12345,12345,12345,12345}; unsigned lfsr113(){ unsigned output; asm("movaps (%1),%%xmm1\n" \ "movaps (%2),%%xmm2\n" \ "movaps (%4),%%xmm0\n" \ "pand %%xmm1,%%xmm2\n" \ "pmulld (%3),%%xmm2\n" \ "pmulld %%xmm1,%%xmm0\n" \ "pxor %%xmm0,%%xmm1\n" \ "psrld \$12,%%xmm1\n" \ "pblendw \$192,%%xmm1,%%xmm3\n" \ "psrld \$1,%%xmm1\n" \ "pblendw \$3,%%xmm1,%%xmm3\n" \ "psrld \$8,%%xmm1\n" \ "pblendw \$48,%%xmm1,%%xmm3\n" \ "psrld \$6,%%xmm1\n" \ "pblendw \$12,%%xmm1,%%xmm3\n" \ "pxor %%xmm2,%%xmm3\n" \ "movaps %%xmm3,(%1)\n" \ "pshufd \$255,%%xmm3,%%xmm0\n" \ "pshufd \$170,%%xmm3,%%xmm1\n" \ "pshufd \$85,%%xmm3,%%xmm2\n" \ "pxor %%xmm0,%%xmm3\n" \ "pxor %%xmm1,%%xmm2\n" \ "pxor %%xmm2,%%xmm3\n" \ "pextrd \$0,%%xmm3,%0\n" \ "":"=&r"(output):"r"(z),"r"(a), "r"(b),"r"(c)); return output; } </pre>	<pre> unsigned z1=12345,z2=12345, z3=12345,z4=12345; unsigned lfsr113() { unsigned long b; b = ((z1 << 6) ^ z1) >> 13; z1 = ((z1 & 4294967294U) << 18) ^ b; b = ((z2 << 2) ^ z2) >> 27; z2 = ((z2 & 4294967288U) << 2) ^ b; b = ((z3 << 13) ^ z3) >> 21; z3 = ((z3 & 4294967280U) << 7) ^ b; b = ((z4 << 3) ^ z4) >> 12; z4 = ((z4 & 4294967168U) << 13) ^ b; return (z1 ^ z2 ^ z3 ^ z4); } </pre>

Table 5: Equivalent realizations for several algorithms for processor with SSE2 support (left column) and in ANSI C language (right column). First row presents calculating four matrix multiplications. Second row presents the packing 16 high bits of 16 integers into one integer. These or similar equivalences are used in constructing the SSE2 algorithms for GM19, GM31 and GM61 algorithms [10].

Four matrix multiplications in parallel, SSE2 version	Same with usual instruction set
<pre> unsigned long x[4],y[4]; [.....] asm("movaps (%0),%%xmm0\n" \ "movaps (%1),%%xmm1\n" \ "padd %xmm1,%%xmm0\n" \ "padd %xmm1,%%xmm0\n" \ "movaps %%xmm0,%%xmm2\n" \ "pslld \$2,%%xmm0\n" \ "padd %xmm1,%%xmm0\n" \ "movaps %%xmm0,(%0)\n" \ "psubd %xmm2,%%xmm0\n" \ "movaps %%xmm0,(%1)\n" \ "":"r"(x),"r"(y)); </pre>	<pre> unsigned long i,newx[4],x[4],y[4]; [.....] for(i=0;i<4;i++){ newx[i]=4*x[i]+9*y[i]; y[i]=3*x[i]+7*y[i]; x[i]=newx[i]; } </pre>
Packing 16 high bits into one integer, SSE2 version	Same with usual instruction set
<pre> unsigned long x[16],output; [.....] asm("movaps (%1),%%xmm0\n" \ "movaps 16(%1),%%xmm1\n" \ "movaps 32(%1),%%xmm2\n" \ "movaps 48(%1),%%xmm3\n" \ "psrld \$31,%%xmm0\n" \ "psrld \$31,%%xmm1\n" \ "psrld \$31,%%xmm2\n" \ "psrld \$31,%%xmm3\n" \ "packssdw %xmm1,%%xmm0\n" \ "packssdw %xmm3,%%xmm2\n" \ "packsswb %xmm2,%%xmm0\n" \ "psllw \$7,%%xmm0\n" \ "pmovmskb %%xmm0,%0\n" \ "":"=r"(output):"r"(x)); </pre>	<pre> const unsigned long halfg=2147483648; unsigned long x[16],i,output=0,bit=1; [.....] for(i=0;i<16;i++){ output+=((x[i]<halfg)?0:bit; bit*=2; } </pre>

The function `mrg32k3a_genPrintState()` prints out the generator state. The function call interface is shown in Table 8.

4.3. Realizations MT19937 and MT19937-SSE

The similar examples as in Table 6 for MT19937 and MT19937-SSE can be found in files `mt19937.c` and `mt19937sse.c` in [10].

The header files for the realizations are `mt19937.h` and `mt19937sse.h`, as it is shown in Table 7.

Table 8 describes the function call interfaces for the functions `mt19937_SetState(...)` and `mt19937sse_SetState(...)` that save a pointer to the corresponding generator state.

Table 9 shows details for the initialization functions `mt19937_genInit(...)` and `mt19937sse_genInit(...)`. Versions of the MT19937 generator are initialized with four unsigned integer values [25].

The functions `mt19937_genRand()` and `mt19937sse_genRand()` generate a pseudorandom 32-bit integer and make respective transformation to the corresponding generator state. Detailed interface for the functions is presented in Table 7.

The function `mt19937sse_genRand624()` generates block of 624 pseudorandom 32-bit unsigned integers and puts them in `state->out[0]`, `state->out[1]`, ..., `state->out[623]`, where `state->out` is array of unsigned integers, and the pointer `state` to the structure of the type `mt19937sse_state` should have previously been saved with the function `mt19937sse_SetState`. Generation of blocks of numbers is further discussed in Sec. 5 and Tables 13 and 14.

The functions `mt19937_genPrintState()` and `mt19937sse_genPrintState()` print out the corresponding generator state. Detailed interface for the functions is shown in Table 8.

4.4. Realizations LFSR113 and LFSR113-SSE

The similar examples as in Table 6 for LFSR113 and LFSR113-SSE can be found in files `lfsr113.c` and `lfsr113sse.c` in [10].

The header files for the realizations are `lfsr113.h` and `lfsr113sse.h`, as it is shown in Table 7.

Table 8 describes the function call interfaces for the functions `lfsr113_SetState(...)` and `lfsr113sse_SetState(...)` that save a pointer to the corresponding generator state.

Table 9 shows details for the initialization functions `lfsr113_genInit(...)` and `lfsr113sse_genInit(...)`. Initializing any version of the generator LFSR113 requires initial values of `z1`, `z2`, `z3`, `z4` described in Sec. 2.3 and Table 4 for the LFSR113 algorithm.

The functions `lfsr113_genRand()` and `lfsr113sse_genRand()` generate a pseudorandom 32-bit integer and make respective transformation to the corresponding generator state. Detailed interface for the functions is presented in Table 7.

The functions `lfsr113_genPrintState()` and `lfsr113sse_genPrintState()` print out the corresponding generator state. Detailed interface for the functions is shown in Table 8.

4.5. Realizations GM19, GM19-SSE, GM31, GM31-SSE, GM61 and GM61-SSE

The similar examples as in Table 6 for GM19, GM19-SSE, GM31, GM31-SSE, GM61 and GM61-SSE can be found in files `gm19.c`, `gm19sse.c`, `gm31.c`, `gm31sse.c`, `gm61.c` and `gm61sse.c` in [10].

The header files for the realizations are `gm19.h`, `gm19sse.h`, `gm31.h`, `gm31sse.h`, `gm61.h` and `gm61sse.h`, as it is shown in Table 7.

Table 8 describes the function call interfaces for the functions `gm19_SetState(...)`, `gm19sse_SetState(...)`, `gm31_SetState(...)`, `gm31sse_SetState(...)`, `gm61_SetState(...)` and `gm61sse_SetState(...)` that save a pointer to the corresponding generator state.

Table 9 shows details for the initialization functions `gm19_genInit(...)`, `gm19sse_genInit(...)`, `gm31_genInit(...)`, `gm31sse_genInit(...)`, `gm61_genInit(...)` and `gm61sse_genInit(...)`. For generators GM19 and GM19SSE it is required that $0 \leq \text{seed} \leq 9$, for GM31 and GM31SSE it is required that $0 \leq \text{seed} \leq 99$, for GM61 and GM61SSE $-0 \leq \text{seed} \leq 999$.

The functions `gm19_genRand()`, `gm19sse_genRand()`, `gm31_genRand()`, `gm31sse_genRand()`, `gm61_genRand()` and `gm61sse_genRand()` generate a pseudorandom 32-bit integer and make respective transformation to the corresponding generator state. Detailed interface for the functions is presented in Table 7.

The functions `gm19_genPrintState()`, `gm19sse_genPrintState()`, `gm31_genPrintState()`, `gm31sse_genPrintState()`, `gm61_genPrintState()` and `gm61sse_genPrintState()`

and

`gm61sse_genPrintState()` print out the corresponding generator state. Detailed interface for the functions is shown in Table 8.

5. Speed and properties

CPU times needed for generating 10^9 random numbers are shown in Tables 10, 11 and 12 for Intel Core 2 Duo E7400, Intel Core i7-940 and AMD Turion X2 RM-70 processors respectively. The results are presented for different compilers and optimization options for all RNGs of our library. We use GNU C compiler gcc version 4.3.3, and Intel C compiler icc version 11.0. In all cases we observe decrease of computation time for the modified algorithms as compared with original algorithms. In particular, using GNU C compiler gcc with optimization level `-O0` we find that MRG32K3ASSE is about 4.5 times faster than MRG32K3A, whereas LFSR113SSE is about 40% faster than LFSR113 and MT19937SSE is about 3 times faster than MT19937. Therefore, the modified generators turn out to be very fast. Algorithms GM19SSE, GM31SSE and GM61SSE are 10–22 times more efficient than corresponding algorithms based on usual command set, and become competitive with other modern generators.

Table 13 contains CPU-times for generating 10^9 random numbers via blocks of numbers, both for our SSE2-based algorithm MT19937SSE and the MT19937 generator from Intel MKL library. The output values of the generator realizations are identical.

Table 14 contains CPU-times for generating 10^9 random numbers via blocks of numbers, both for our SSE2-based algorithm MRG32K3ASSE and the MRG32K3A generator from Intel MKL library. The output values of the generator realizations are identical.

6. Statistical properties and period lengths

Applying hundreds of empirical tests (so-called batteries of tests) allows to estimate statistical robustness of the generators. As we mentioned above, here we considered only generators with best statistical properties (see also Sec. 4.5.4 and Sec. 4.6.1 in [19], and Sec. 3 in [6]).

Table 15 shows the results of applying the SmallCrush, PseudoDiehard, Crush and Bigcrush batteries of tests taken from [26], to the generators of our

library. For each battery of tests, Table 15 shows three numbers: the number of statistical tests with p-values outside the interval $[10^{-3}, 1 - 10^{-3}]$, number of tests with p-values outside the interval $[10^{-5}, 1 - 10^{-5}]$, and number of tests with p-values outside the interval $[10^{-10}, 1 - 10^{-10}]$. The last column in Table 15 shows the RNG period lengths.

Libraries SmallCrush, PseudoDiehard, Crush and Bigcrush contain 15, 126, 144 and 160 tests respectively. We consider the statistical test “failed” if the p-value lies outside the region $[10^{-3}, 1 - 10^{-3}]$. Varying free initial conditions of the generators GM19, GM31 and GM61 and applying statistical tests, one can observe occasionally a single failed test with p-value in the region $[10^{-5}, 10^{-3}] \cup [1 - 10^{-3}, 1 - 10^{-5}]$. Since the number of applied tests is of the order of 10^3 , such a single failed test can be explained simply by random statistical fluctuations, which do not characterize the quality of random numbers. As seen, all generators possess excellent statistical properties. Table 15 shows that the generators, except LFSR113 and MT19937 pass all statistical tests in the TestU01 library. Such generators can be recommended for practical use.

As was mentioned in [27], the generators MT19937 and LFSR113 fail tests that try to find a linear structure in the bits of the output, because their bits have a linear structure by construction. Hence this particular point does not mean serious deficiencies of the generator MT19937. MT19937 passes all other tests. Namely, both of MT19937 and LFSR113 fail the test `scomp_LinearComp`, and also LFSR113 fails the test `smarsa_MatrixRank`.

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Table 7: Function call interfaces for generating 32-bit pseudorandom numbers.

Generator name	File to include	State type	Generate 32-bit random number(s)
GM19	gm19.h	gm19_state	unsigned gm19_genRand();
GM19 (SSE)	gm19sse.h	gm19sse_state	unsigned gm19sse_genRand();
GM31	gm31.h	gm31_state	unsigned gm31_genRand();
GM31 (SSE)	gm31sse.h	gm31sse_state	unsigned gm31sse_genRand();
GM61	gm61.h	gm61_state	unsigned gm61_genRand();
GM61 (SSE)	gm61sse.h	gm61sse_state	unsigned gm61sse_genRand();
LSFR113	lfsr113.h	lfsr113_state	unsigned lfsr113_genRand();
LSFR113 (SSE)	lfsr113sse.h	lfsr113sse_state	unsigned lfsr113sse_genRand();
MRG32K3A	mrg32k3a.h	mrg32k3a_state	double mrg32k3a_genRand();
MRG32K3A (SSE)	mrg32k3asse.h	mrg32k3asse_state	unsigned mrg32k3asse_genRand();
MRG32K3A (SSE,4)	mrg32k3asse.h	mrg32k3asse_state	void mrg32k3asse_genRand4(unsigned* arr);
MRG32K3A (SSE,32)	mrg32k3asse32.h	mrg32k3asse32_state	void mrg32k3asse32_genRand32(unsigned* arr);
MT19937	mt19937.h	mt19937_state	unsigned long mt19937_genRand();
MT19937 (SSE)	mt19937sse.h	mt19937sse_state	unsigned mt19937sse_genRand();
MT19937 (SSE,624)	mt19937sse.h	mt19937sse_state	void mt19937sse_genRand624();

Table 8: Function call interfaces for keeping the pointer to a state and for printing out a generator state.

Set state	Print out state
void gm19_SetState(gm19_state* state);	void gm19_genPrintState();
void gm19sse_SetState(gm19sse_state* state);	void gm19sse_genPrintState();
void gm31_SetState(gm31_state* state);	void gm31_genPrintState();
void gm31sse_SetState(gm31sse_state* state);	void gm31sse_genPrintState();
void gm61_SetState(gm61_state* state);	void gm61_genPrintState();
void gm61sse_SetState(gm61sse_state* state);	void gm61sse_genPrintState();
void lfsr113_SetState(lfsr113_state* state);	void lfsr113_genPrintState();
void lfsr113sse_SetState(lfsr113sse_state* state);	void lfsr113sse_genPrintState();
void mrg32k3a_SetState(mrg32k3a_state* state);	void mrg32k3a_genPrintState();
void mrg32k3asse_SetState(mrg32k3asse_state* state);	void mrg32k3asse_genPrintState();
void mrg32k3asse32_SetState(mrg32k3asse32_state* state);	void mrg32k3asse32_genPrintState();
void mt19937_SetState(mt19937_state* state);	void mt19937_genPrintState();
void mt19937sse_SetState(mt19937sse_state* state);	void mt19937sse_genPrintState();

Table 9: Function call interfaces for initializing the generators.

State type	Initialize state
gm19_state	void gm19_genInit(unsigned seed);
gm19sse_state	void gm19sse_genInit(unsigned seed);
gm31_state	void gm31_genInit(unsigned seed);
gm31sse_state	void gm31sse_genInit(unsigned seed);
gm61_state	void gm61_genInit(unsigned seed);
gm61sse_state	void gm61sse_genInit(unsigned seed);
lfsr113_state	void lfsr113_genInit(unsigned z1,unsigned z2,unsigned z3,unsigned z4);
lfsr113sse_state	void lfsr113sse_genInit(unsigned z1,unsigned z2,unsigned z3,unsigned z4);
mrg32k3a_state	void mrg32k3a_genInit(double x0,double x1,double x2, double y0,double y1,double y2);
mrg32k3asse_state	void mrg32k3asse_genInit(unsigned x0,unsigned x1,unsigned x2, unsigned y0,unsigned y1,unsigned y2);
mrg32k3asse32_state	void mrg32k3asse32_genInit(unsigned x0,unsigned x1,unsigned x2, unsigned y0,unsigned y1,unsigned y2);
mt19937_state	void mt19937_genInit(unsigned long init0, unsigned long init1, unsigned long init2, unsigned long init3);
mt19937sse_state	void mt19937sse_genInit(unsigned long init0, unsigned long init1, unsigned long init2, unsigned long init3);

Table 10: CPU time (sec) for generating 10^9 random numbers. Processor: Intel Core 2 Duo E7400. Compilers: gcc 4.3.3, icc 11.0.

	gcc -O0	gcc -O1	gcc -O2	gcc -O3	icc -O0	icc -O1	icc -O2	icc -O3
MRG32k3a	47.4	35.7	36.0	27.2	55.9	32.6	26.0	26.0
MRG32k3aSSE	10.5	8.8	7.2	7.2	10.4	8.7	7.4	7.7
MRG32k3a-4SSE	7.3	6.9	6.9	6.9	7.3	6.9	6.9	6.9
LFSR113	10.9	4.6	4.8	3.5	10.8	5.1	4.6	4.6
LFSR113SSE	8.0	7.4	7.3	7.3	8.3	7.7	7.5	7.5
MT19937	16.6	6.1	6.1	2.7	15.8	6.8	2.9	2.8
MT19937SSE	5.8	5.2	6.1	2.3	5.9	5.3	2.3	2.3
GM19	616.9	242.2	200.1	144.4	628.9	226.9	125.1	124.9
GM19SSE	32.8	30.6	27.8	27.7	33.5	29.8	28.5	28.2
GM31	1067.9	255.6	204.9	151.4	1110.5	318.0	175.8	183.0
GM31SSE	49.9	45.5	41.8	41.8	49.5	44.4	43.2	45.0
GM61	1794.0	987.5	923.6	844.5	1908.9	1002.1	853.4	864.4
GM61SSE	184.5	183.0	181.6	181.6	186.0	192.3	181.8	188.7

Table 11: CPU time (sec) for generating 10^9 random numbers. Processor: Intel Core i7-940. Compilers: gcc 4.3.3, icc 11.0.

	gcc -O0	gcc -O1	gcc -O2	gcc -O3	icc -O0	icc -O1	icc -O2	icc -O3
MRG32k3a	47.9	36.3	35.3	25.0	56.1	33.1	22.8	28.1
MRG32k3aSSE	9.1	7.4	5.8	5.8	8.8	7.4	6.0	5.9
MRG32k3a-4SSE	6.2	5.8	5.8	5.7	6.4	5.9	5.9	5.7
LFSR113	10.4	4.8	6.8	3.1	10.2	5.0	4.6	4.5
LFSR113SSE	8.0	6.8	6.8	6.9	7.3	6.9	6.6	6.5
MT19937	13.7	5.7	6.9	2.6	17.5	6.5	2.9	2.9
MT19937SSE	5.2	4.8	5.5	2.0	4.9	4.7	2.4	2.0
GM19	578.5	210.1	163.1	117.1	604.3	259.4	110.7	123.3
GM19SSE	32.3	28.7	26.7	29.9	33.6	33.0	27.8	34.1
GM31	999.0	244.6	181.6	134.3	978.9	298.8	143.4	151.0
GM31SSE	39.1	36.4	38.5	58.7	47.1	36.4	35.4	35.9
GM61	1599.7	893.8	836.9	766.6	1606.5	870.8	770.1	795.1
GM61SSE	120.6	123.0	116.8	117.2	124.6	120.2	130.7	117.7

Table 12: CPU time (sec) for generating 10^9 random numbers. Processor: AMD Turion X2 RM-70. Compilers: gcc 4.3.3, icc 11.0.

	gcc -O0	gcc -O1	gcc -O2	gcc -O3	icc -O0	icc -O1	icc -O2	icc -O3
MRG32k3a	89.0	60.9	60.9	47.0	89.1	69.2	41.5	41.6
MRG32k3aSSE	25.9	22.3	18.4	18.3	25.6	22.3	19.0	19.0
MRG32k3a-4SSE	18.3	18.2	18.2	18.2	18.1	18.2	18.1	18.3
LFSR113	14.6	8.7	9.6	5.3	14.9	9.1	6.9	6.8
MT19937	31.0	17.8	10.8	7.1	31.0	18.7	5.2	4.9
MT19937SSE	11.3	10.3	11.1	6.6	10.8	9.9	6.0	6.0
GM19	1651.5	385.1	259.9	222.1	1759.9	375.4	198.6	198.4
GM19SSE	96.0	105.3	87.5	87.5	103.5	102.8	88.6	88.9
GM31	1512.5	436.7	313.2	237.3	1573.8	506.0	228.7	243.9
GM31SSE	136.9	130.0	130.0	131.1	137.1	130.9	128.3	128.5
GM61	5138.9	3826.0	3991.9	3567.9	5221.6	3849.2	3604.7	3586.9
GM61SSE	418.3	413.6	411.1	409.7	427.5	416.6	407.7	410.2

Table 13: Comparing CPU-times for different versions of MT19937 for generating 10^9 random numbers via blocks of N numbers. Speed is also compared with the respective realization of MT19937 generator in Intel MKL library. Processor: Intel Core i7-940.

N	Intel MKL library				Our version			
	icc -O0	icc -O1	icc -O2	icc -O3	icc -O0	icc -O1	icc -O2	icc -O3
1	24.7	22.3	24.0	22.1	4.9	4.7	2.4	2.0
624	1.8	1.8	1.8	1.8	1.8	1.6	1.4	1.4

Table 14: Comparing CPU-times for different versions of MRG32K3A for generating 10^9 random numbers via blocks of N numbers. Speed is also compared with the respective realization of MRG32K3A generator in Intel MKL library. Processors: Intel Xeon 5160 (upper table), Intel Core i7-940 (lower table).

N	Intel MKL library				Our version			
	icc -O0	icc -O1	icc -O2	icc -O3	icc -O0	icc -O1	icc -O2	icc -O3
1	37.5	33.9	33.0	33.0	10.0	8.2	7.2	7.2
4	18.4	18.4	17.8	17.8	7.3	6.6	6.6	6.6
32	10.6	10.6	10.6	10.6	6.6	6.6	6.5	6.5
1000	6.4	6.6	6.6	6.6	6.6	6.6	6.6	6.6

N	Intel MKL library				Our version			
	icc -O0	icc -O1	icc -O2	icc -O3	icc -O0	icc -O1	icc -O2	icc -O3
1	31.2	27.7	28.1	29.6	8.8	7.4	6.0	5.9
4	16.3	16.0	15.6	15.7	6.4	5.9	5.9	5.7
32	8.6	8.6	8.5	8.8	5.4	5.3	5.2	5.2
1000	5.3	5.3	5.1	5.1	5.5	5.3	5.3	5.3

Table 15: Properties of the generators: numbers of failed tests for the batteries of tests SmallCrush, Crush, Bigcrush [26], and Diehard [26], and period lengths. Testing was performed with package TestU01 version TestU01-1.2.3. For each battery of tests, we present three numbers: the number of statistical tests with p-values outside the interval $[10^{-3}, 1 - 10^{-3}]$, number of tests with p-values outside the interval $[10^{-5}, 1 - 10^{-5}]$, and number of tests with p-values outside the interval $[10^{-10}, 1 - 10^{-10}]$.

Generator	SmallCrush	Diehard	Crush	Bigcrush	Period
MRG32k3a	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 0	$3.1 \cdot 10^{57}$
LFSR113	0, 0, 0	1, 0, 0	6, 6, 6	6, 6, 6	$1.0 \cdot 10^{34}$
MT19937	0, 0, 0	0, 0, 0	2, 2, 2	2, 2, 2	$4.3 \cdot 10^{6001}$
GM19SSE	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 0	$2.7 \cdot 10^{11}$
GM31SSE	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 0	$4.6 \cdot 10^{18}$
GM61SSE	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 0	$5.3 \cdot 10^{36}$

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